THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 I/J University Mathematics 2015-2016
Assignment 1

1. Find the following limits.

   (a) $\lim_{n\to\infty} n\left(\sqrt{1+\frac{1}{n}} - \sqrt{1-\frac{1}{n}}\right)$

   (b) $\lim_{n\to\infty} \frac{n\sin(e^n)}{n^2+3}$

2. (a) If $\frac{x+4}{x^2+3x+2} \equiv \frac{A}{x+2} + \frac{B}{x+1}$ for some real numbers $A$ and $B$, find the values of $A$ and $B$.

   (b) By using the result in (a), find $\sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{k+4}{k^2+3k+2}\right)$.

   (i.e. $\lim_{n\to\infty}\sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{k+4}{k^2+3k+2}\right)$)

3. By using the sandwich theorem, prove that

$$\lim_{n\to\infty}\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2}\right) = 0.$$

4. Let $a$ and $b$ be two positive real numbers and let $\{x_n\}$ be a sequence of positive real numbers such that

$$0 < x_1 < b \qquad \text{and} \qquad x_{n+1} = \sqrt{\frac{ab^2 + x_n^2}{a+1}} \text{ for } n \geq 1.$$

   (a) Prove that $\{x_n\}$ is monotonic increasing.

   (b) Prove that $\{x_n\}$ converges (i.e. $\lim_{n\to\infty} x_n$ exists) and hence find its limit.

5. Let $\{x_n\}$ and $\{y_n\}$ be sequences of positive real numbers such that $0 < y_1 \leq x_1$ and for $n = 1, 2, 3, \cdots$

$$x_{n+1} = \frac{x_n + y_n}{2} \qquad \text{and} \qquad y_{n+1} = \frac{2x_n y_n}{x_n + y_n}.$$

   (a) Show that $x_n \geq y_n$ for all natural numbers $n$.

   (b) Prove that $\{x_n\}$ is a monotonic decreasing sequence and $\{y_n\}$ is a monotonic increasing sequence.

   (c) Prove that $\{x_n\}$ and $\{y_n\}$ converge and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

   (d) Prove that $x_n y_n$ is a constant and hence find $\lim_{n\to\infty} x_n$.

6. Let $\{a_n\}$ be a sequence of real numbers defined by $a_n = \left(1 + \frac{1}{n}\right)^n$ for $n = 1, 2, 3, \cdots$.

   (a) By using the binomial theorem, show that when $n \geq 2$,

$$a_n = 2 + \sum_{r=2}^{n}\frac{1}{r!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{r-1}{n}\right).$$

   Hence, show that $a_{n+1} \geq a_n$ for $n \geq 2$.

(b) By using the inequality in (a) and considering the inequality $\frac{1}{r!} \leq \frac{1}{2^r - 1}$, show that when $n \geq 2$, $a_n \leq 3$.

(c) Show that $\lim_{n \to \infty} a_n$ exists.